

# Remarks on the Aharonov-Casher dynamics in a CPT-odd Lorentz-violating background

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**Abstract** – The Aharonov-Casher problem in the presence of a Lorentz-violating background nonminimally coupled to a spinor and a gauge field is examined. Using an approach based on the self-adjoint extension method, an expression for the bound state energies is obtained in terms of the physics of the problem by determining the self-adjoint extension parameter.

Since the construction of the standard model extension (SME), proposed by Colladay and Kostelecký [1–3] (see also [4–6]) quantum field theory systems have been studied in the presence of Lorentz symmetry violation. The SME includes Lorentz-violating (LV) terms in all the sectors of the minimal standard model, becoming a suitable tool to address LV effects in distinct physical systems. Several investigations have been developed in the context of this theoretical framework in the latest years, involving field theories [7–23], aspects on the gauge sector of the SME [24–30], quantum electrodynamics [31–36], and astrophysics [37–39]. These many contributions have elucidated the effects induced by Lorentz violation and served to set up stringent upper bounds on the LV coefficients [40].

Another way to propose and investigate Lorentz violation is considering new interaction terms. In particular, in ref. [41], a Lorentz-violating and CPT-odd nonminimal coupling between fermions and the gauge field was firstly proposed in the form

$$D_\mu = \partial_\mu + ieA_\mu + i\frac{g}{2}\epsilon_{\mu\lambda\alpha\nu}V^\lambda F^{\alpha\nu}, \quad (1)$$

in the context of the Dirac equation,  $(i\gamma^\mu D_\mu - m)\Psi = 0$ . In this case, the fermion spinor is  $\Psi$ , while  $V^\mu = (V_0, \mathbf{V})$  is the Carroll-Field-Jackiw four-vector, and  $g$  is the constant that measures the nonminimal coupling magnitude. The analysis of the nonrelativistic limit of Eq. (1) revealed that this nonminimal coupling generates a magnetic dipole

moment ( $g\mathbf{V}$ ) even for uncharged particles [41], yielding an Aharonov-Casher (AC) phase for its wave function. In these works, after assessing the nonrelativistic regime, one has identified a generalized canonical momentum,

$$\boldsymbol{\pi} = \mathbf{p} - e\mathbf{A} + gV_0\mathbf{B} - g\mathbf{V} \times \mathbf{E}, \quad (2)$$

which allows to introduce this nonminimal coupling in an operational way, *i.e.*, just redefining the vector potential and the corresponding magnetic field as indicated below:

$$\mathbf{A} \rightarrow \mathbf{A} + \frac{g}{e}(\mathbf{V} \times \mathbf{E}), \quad (3)$$

$$\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A} \rightarrow \boldsymbol{\nabla} \times \mathbf{A} - \frac{g}{e}\boldsymbol{\nabla} \times (\mathbf{V} \times \mathbf{E}). \quad (4)$$

This CPT-odd nonminimal coupling was further analyzed in various contexts in relativistic quantum mechanics [42–50].

The aim of the present work is to study the effect of this CPT-odd LV nonminimal interaction on the AC dynamics. Taking into account Eqs. (3) and (4) we obtain the Schrödinger-Pauli equation

$$\hat{H}\Psi = E\Psi, \quad (5)$$

where

$$\hat{H} = \frac{1}{2M} \left\{ \left[ \mathbf{p} - e\left(\mathbf{A} + \frac{g}{e}\mathbf{V} \times \mathbf{E}\right) \right]^2 + eU(r) - e\boldsymbol{\sigma} \cdot \left[ \boldsymbol{\nabla} \times \mathbf{A} - \frac{g}{e}\boldsymbol{\nabla} \times (\mathbf{V} \times \mathbf{E}) \right] \right\}, \quad (6)$$

is the Hamiltonian operator. As is well-known, the potential  $U(r)$ , in cylindrical coordinates, is given by

$$U(r) = -\phi \ln \left( \frac{r}{r_0} \right). \quad (7)$$

It is very difficult to solve Eq. (5) by including both effects taking into account the logarithmic form of  $U(r)$ . This potential is relevant if we consider the full Hamiltonian, which is compatible with a charged solenoid. While the AC effect stems from the quantity  $\boldsymbol{\sigma} \cdot [g \nabla \times (\mathbf{V} \times \mathbf{E})]$ , one can affirm that the LV background does not contribute to the Aharonov-Bohm (AB) effect. To solve Eq. (5) considering both effects (AB and AC) such potential has to be regarded. Since we are interested only in the background effects, we can examine only the sector generating the AC effect. In this latter case, the field configuration is given by

$$\mathbf{E} = \phi \frac{\hat{\mathbf{r}}}{r}, \quad \nabla \cdot \mathbf{E} = \phi \frac{\delta(r)}{r}, \quad \phi = \frac{\lambda}{2\pi\epsilon_0}, \quad \mathbf{V} = V \hat{\mathbf{z}}, \quad (8)$$

where  $\mathbf{E}$  is the electric field generated by an infinite charge filament and  $\lambda$  is the charge density along the  $z$ -axis. After this identification, the Hamiltonian (6) becomes

$$\hat{H} = \frac{1}{2M} \left[ \left( \frac{1}{i} \nabla - \nu \frac{\hat{\mathbf{r}}}{r} \right)^2 + \nu \sigma_z \frac{\delta(r)}{r} \right], \quad (9)$$

with

$$\nu = gV\phi, \quad (10)$$

the coupling constant of the  $\delta(r)/r$  potential. Here, we are only interested in the situation in which  $\hat{H}$  possesses bound states.

The Hamiltonian in Eq. (9) governs the quantum dynamics of a spin-1/2 neutral particle with a radial electric field, *i.e.*, a spin-1/2 AC problem, with  $g\mathbf{V}$  playing the role of a nontrivial magnetic dipole moment. Note the presence of a  $\delta$  function singularity at the origin in Eq. (9) which turns it more complicated to be solved. Such kind of point interaction potential can then be addressed by the self-adjoint extension approach [51], used here for determining the bound states.

Making use of the underlying rotational symmetry expressed by the fact that  $[\hat{H}, \hat{J}_z] = 0$ , where  $\hat{J}_z = -i\partial/\partial\phi + \sigma_z/2$  is the total angular momentum operator in the  $z$ -direction, we decompose the Hilbert space  $\mathfrak{H} = L^2(\mathbb{R}^2)$  with respect to the angular momentum  $\mathfrak{H} = \mathfrak{H}_r \otimes \mathfrak{H}_\phi$ , where  $\mathfrak{H}_r = L^2(\mathbb{R}^+, r dr)$  and  $\mathfrak{H}_\phi = L^2(\mathcal{S}^1, d\phi)$ , with  $\mathcal{S}^1$  denoting the unit sphere in  $\mathbb{R}^2$ . So it is possible to express the eigenfunctions of the two dimensional Hamiltonian in terms of the eigenfunctions of  $\hat{J}_z$ :

$$\Psi(r, \phi) = \begin{pmatrix} \psi_m(r) e^{i(m_j - 1/2)\phi} \\ \chi_m(r) e^{i(m_j + 1/2)\phi} \end{pmatrix}, \quad (11)$$

with  $m_j = m + 1/2 = \pm 1/2, \pm 3/2, \dots$ , with  $m \in \mathbb{Z}$ . By inserting Eq. (11) into Eq. (5) the Schrödinger-Pauli equation for  $\psi_m(r)$  is found to be

$$H\psi_m(r) = E\psi_m(r), \quad (12)$$

where

$$H = H_0 + \frac{\nu}{2M} \frac{\delta(r)}{r}, \quad (13)$$

and

$$H_0 = -\frac{1}{2M} \left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{(m - \nu)^2}{r^2} \right]. \quad (14)$$

An operator  $\mathcal{O}$ , with domain  $\mathcal{D}(\mathcal{O})$ , is said to be self-adjoint if and only if  $\mathcal{D}(\mathcal{O}^\dagger) = \mathcal{D}(\mathcal{O})$  and  $\mathcal{O}^\dagger = \mathcal{O}$ . For smooth functions,  $\xi \in C_0^\infty(\mathbb{R}^2)$  with  $\xi(0) = 0$ , we should have  $H\xi = H_0\xi$ . Hence, it is reasonable to interpret the Hamiltonian (13) as a self-adjoint extension of  $H_0|_{C_0^\infty(\mathbb{R}^2 \setminus \{0\})}$  [52–54]. It is a well-known fact that the symmetric radial operator  $H_0$  is essentially self-adjoint for  $|m - \nu| \geq 1$  [55]. For those values of the  $m$  fulfilling  $|m - \nu| < 1$  it is not essentially self-adjoint, admitting an one-parameter family of self-adjoint extensions,  $H_{\theta,0}$ , where  $\theta \in [0, 2\pi)$  is the self-adjoint extension parameter. To characterize this family and determine the bound state energy, we will follow a general approach proposed in ref. [56] (cf. also refs. [57–60]), which is based on the boundary conditions that hold at the origin [61]. The boundary condition is a match of the logarithmic derivatives of the zero-energy ( $E = 0$ ) solutions for Eq. (12) and the solutions for the problem defined by  $H_0$  plus the self-adjoint extension.

Now, the goal is to find the bound states for the Hamiltonian (13). Then, we temporarily forget the  $\delta$  function potential and find the boundary conditions allowed for  $H_0$ . However, the self-adjoint extension provides an infinity of possible boundary conditions, and it can not give us the true physics of the problem. Nevertheless, once the physics at  $r = 0$  is known [57,58], it is possible to determine any arbitrary parameter coming from the self-adjoint extension, and then we have a complete description of the problem. Since we have a singular point we must guarantee that the Hamiltonian is self-adjoint in the region of motion.

One observes that even if the operator is Hermitian  $H_0^\dagger = H_0$ , its domains could be different. The self-adjoint extension approach consists, essentially, in extending the domain  $\mathcal{D}(H_0)$  to match  $\mathcal{D}(H_0^\dagger)$ , therefore turning  $H_0$  self-adjoint. To do this, we must find the deficiency subspaces,  $N_\pm$ , with dimensions  $n_\pm$ , which are called deficiency indices of  $H_0$  [55]. A necessary and sufficient condition for  $H_0$  being essentially self-adjoint is that  $n_+ = n_- = 0$ . On the other hand, if  $n_+ = n_- \geq 1$ , then  $H_0$  has an infinite number of self-adjoint extensions parametrized by unitary operators  $U : N_+ \rightarrow N_-$ . In order to find the deficiency subspaces of  $H_0$  in  $\mathfrak{H}_r$ , we must solve the eigenvalue equation

$$H_0^\dagger \psi_\pm = \pm i k_0 \psi_\pm, \quad (15)$$

where  $k_0 \in \mathbb{R}$  is introduced for dimensional reasons. Since  $H_0^\dagger = H_0$ , the only square-integrable functions which are solutions of Eq. (15) are the modified Bessel functions of second kind,

$$\psi_\pm = K_{|m-\nu|}(r\sqrt{\mp\epsilon}), \quad (16)$$

with  $\varepsilon = 2iMk_0$ . These functions are square integrable only in the range  $|m - \nu| < 1$ , for which  $H_0$  is not self-adjoint. The dimension of such deficiency subspace is  $(n_+, n_-) = (1, 1)$ . According to the von Neumann-Krein theory, the domain of  $H_{\theta,0}$  is given by

$$\mathcal{D}(H_{\theta,0}) = \mathcal{D}(H_0^\dagger) = \mathcal{D}(H_0) \oplus N_+ \oplus N_- \quad (17)$$

Thus,  $\mathcal{D}(H_{\theta,0})$  in  $\mathfrak{H}_r$  is given by the set of functions [55]

$$\psi_\theta(r) = \psi_m(r) + c [K_{|m-\nu|}(r\sqrt{-\varepsilon}) + e^{i\theta} K_{|m-\nu|}(r\sqrt{\varepsilon})], \quad (18)$$

where  $\psi_m(r)$ , with  $\psi_m(0) = \dot{\psi}_m(0) = 0$  ( $\dot{\psi} \equiv d\psi/dr$ ), is a regular wave function and  $\theta \in [0, 2\pi)$  represents a choice for the boundary condition.

Now, we are in position to determine a fitting value for  $\theta$ . To do so, we follow the approach of ref. [61]. First, one considers the zero-energy solutions  $\psi_0$  and  $\psi_{\theta,0}$  for  $H$  and  $H_0$ , respectively, *i.e.*,

$$\left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{(m-\nu)^2}{r^2} - \nu \frac{\delta(r)}{r} \right] \psi_0 = 0, \quad (19)$$

and

$$\left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{(m-\nu)^2}{r^2} \right] \psi_{\theta,0} = 0. \quad (20)$$

The value of  $\theta$  is determined by the boundary condition

$$\lim_{a \rightarrow 0^+} \left( a \frac{\dot{\psi}_0}{\psi_0} \Big|_{r=a} - a \frac{\dot{\psi}_{\theta,0}}{\psi_{\theta,0}} \Big|_{r=a} \right) = 0. \quad (21)$$

The first term of (21) is obtained by integrating (19) from 0 to  $a$ . The second term is calculated using the asymptotic representation for the Bessel function  $K_{|m-\nu|}$  for small argument. So, from (21) we arrive at

$$\frac{\dot{\Upsilon}_\theta(a)}{\Upsilon_\theta(a)} = \nu, \quad (22)$$

with

$$\Upsilon_\theta(r) = D(-\varepsilon) + e^{i\theta} D(+\varepsilon), \quad (23)$$

and

$$D(\pm\varepsilon) = \frac{(r\sqrt{\pm\varepsilon})^{-|m-\nu|}}{2^{-|m-\nu|}\Gamma(1-|m-\nu|)} - \frac{(r\sqrt{\pm\varepsilon})^{|m-\nu|}}{2^{|m-\nu|}\Gamma(1+|m-\nu|)}. \quad (24)$$

Eq. (22) gives us the parameter  $\theta$  in terms of the physics of the problem, *i.e.*, the correct behavior of the wave functions at the origin.

Next, we will find the bound states of the Hamiltonian  $H_0$  and, by using (22), the spectrum of  $H$  will be determined without any arbitrary parameter. Then, from  $H_0\psi_\theta = E\psi_\theta$  we achieve the modified Bessel equation ( $\kappa^2 = -2ME$ )

$$\left[ \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{(m-\nu)^2}{r^2} - \kappa^2 \right] \psi_\theta(r) = 0, \quad (25)$$

where  $E < 0$  (since we are looking for bound states). The general solution for the above equation is

$$\psi_\theta(r) = K_{|m-\nu|}(r\sqrt{-2ME}). \quad (26)$$

Since these solutions belong to  $\mathcal{D}(H_{\theta,0})$ , they present the form (18) for a  $\theta$  selected from the physics of the problem (cf. Eq. (22)). So, we substitute (26) into (18) and compute  $a\psi_\theta/\psi_\theta|_{r=a}$ . After a straightforward calculation, we have the relation

$$\frac{|m-\nu| [a^{2|m-\nu|}(-ME)^{|m-\nu|}\Theta - 1]}{a^{2|m-\nu|}(-ME)^{|m-\nu|}\Theta + 1} = \nu, \quad (27)$$

where  $\Theta = \Gamma(-|m-\nu|)/(2^{|m-\nu|}\Gamma(|m-\nu|))$ . Solving the above equation for  $E$ , we find the sought energy spectrum

$$E = -\frac{2}{Ma^2} \left[ \left( \frac{\nu + |m-\nu|}{\nu - |m-\nu|} \right) \frac{\Gamma(1+|m-\nu|)}{\Gamma(1-|m-\nu|)} \right]^{1/|m-\nu|}. \quad (28)$$

In the above relation, to ensure that the energy is a real number, we must have  $|\nu| \geq |m-\nu|$ , and due to  $|m-\nu| < 1$  it is sufficient to consider  $|\nu| \geq 1$ . A necessary condition for a  $\delta$  function generating an attractive potential, able to support bound states, is that the coupling constant must be negative. Thus, the existence of bound states with real energies requires

$$\nu \leq -1. \quad (29)$$

From the above equation and Eq (10) it follows that  $gV\lambda < 0$ , and there is a minimum value for this product.

In conclusion, we have analyzed the effects of a LV background vector, nonminimally coupled to the gauge and fermion fields, on the AC problem. The self-adjoint extension approach was used to determine the bound states of the particle in terms of the physics of the problem, in a very consistent way and without any arbitrary parameter.

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